ASU Sound & Vibration Lab

Faculty of Engineering
Ain Shams University

Acoustics

Vibration of Continuous Systems and Wave Equation

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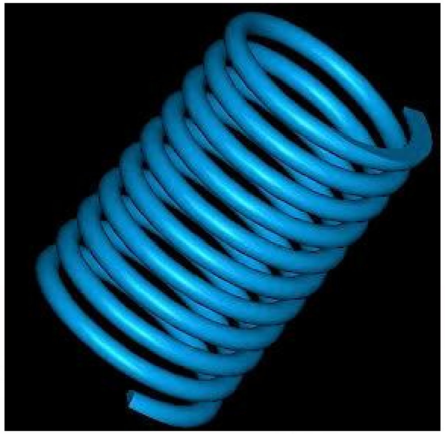
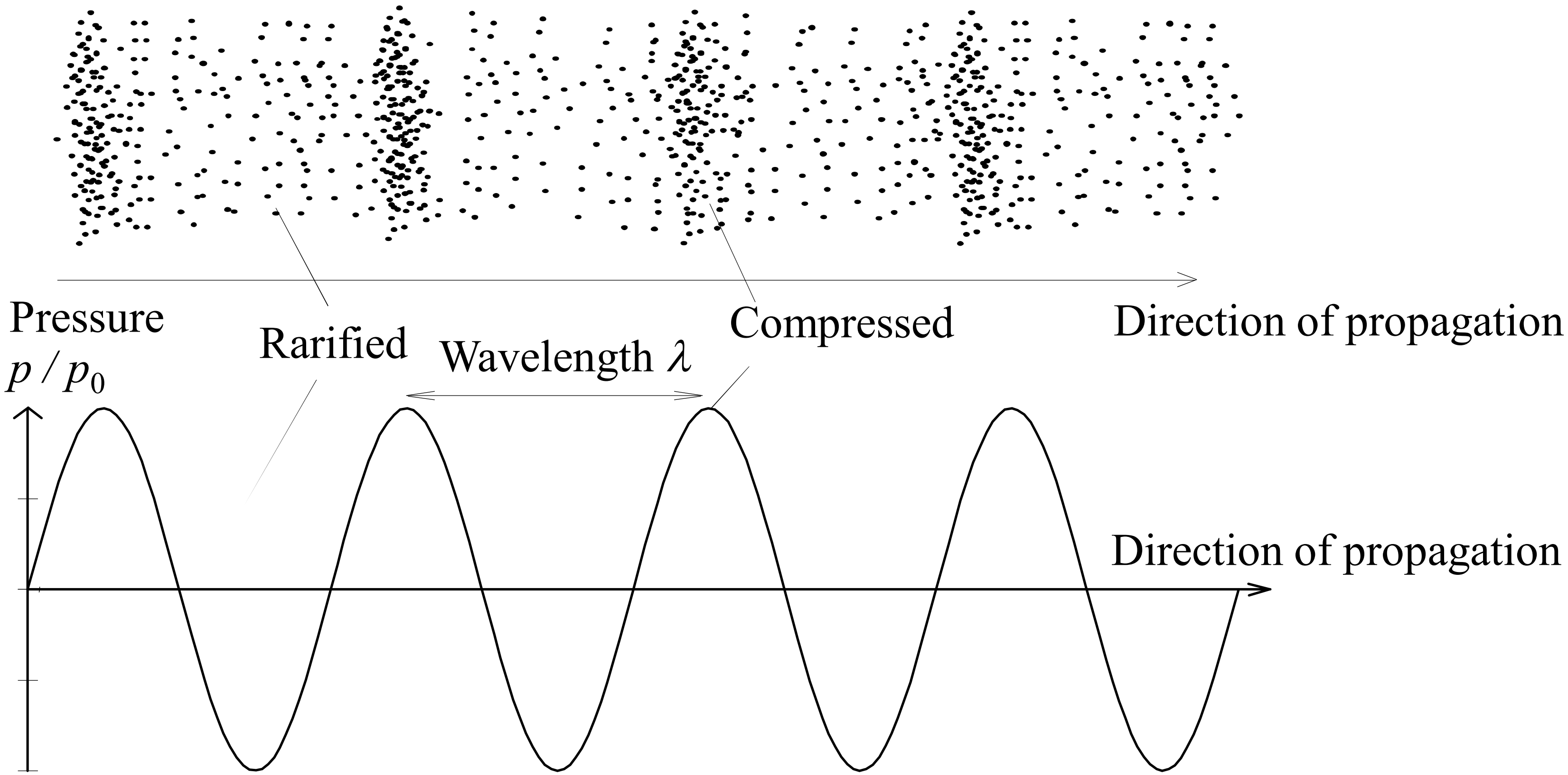
#6: Wave Equation and Measurement Techniques

Continuous Systems

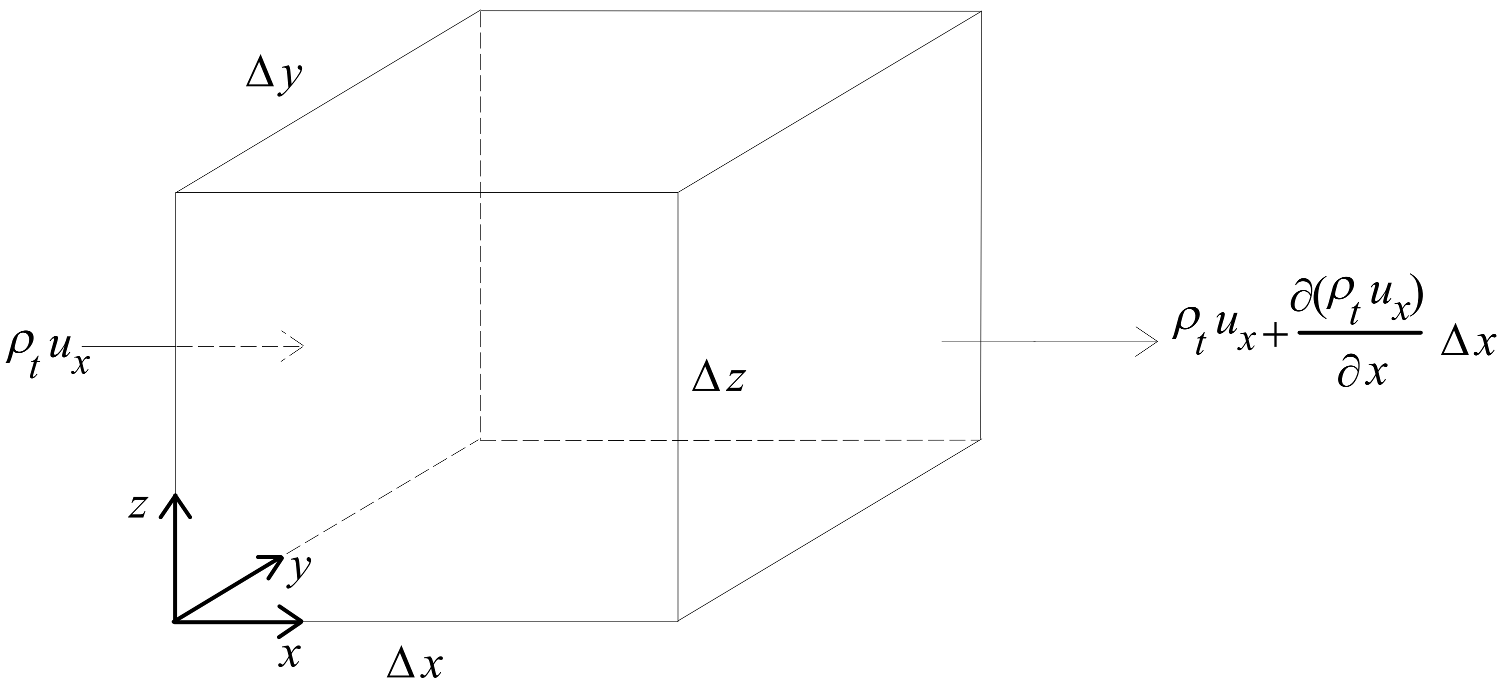


The Wave Equation in Fluids

Wave Equation in Fluids

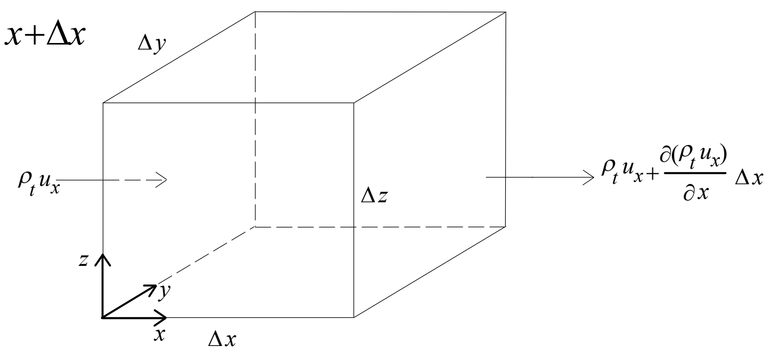


Equation of Continuity



Equation of Continuity

$$\frac{\partial}{\partial t}(\rho_t \Delta x \Delta y \Delta z) = (\rho_t u_x \Delta y \Delta z)_x - (\rho_t u_x \Delta y \Delta z)_{x+\Delta x}$$



$$\frac{\partial}{\partial t}(\rho_t \Delta x \Delta y \Delta z) = (\rho_t u_x \Delta y \Delta z)_x$$

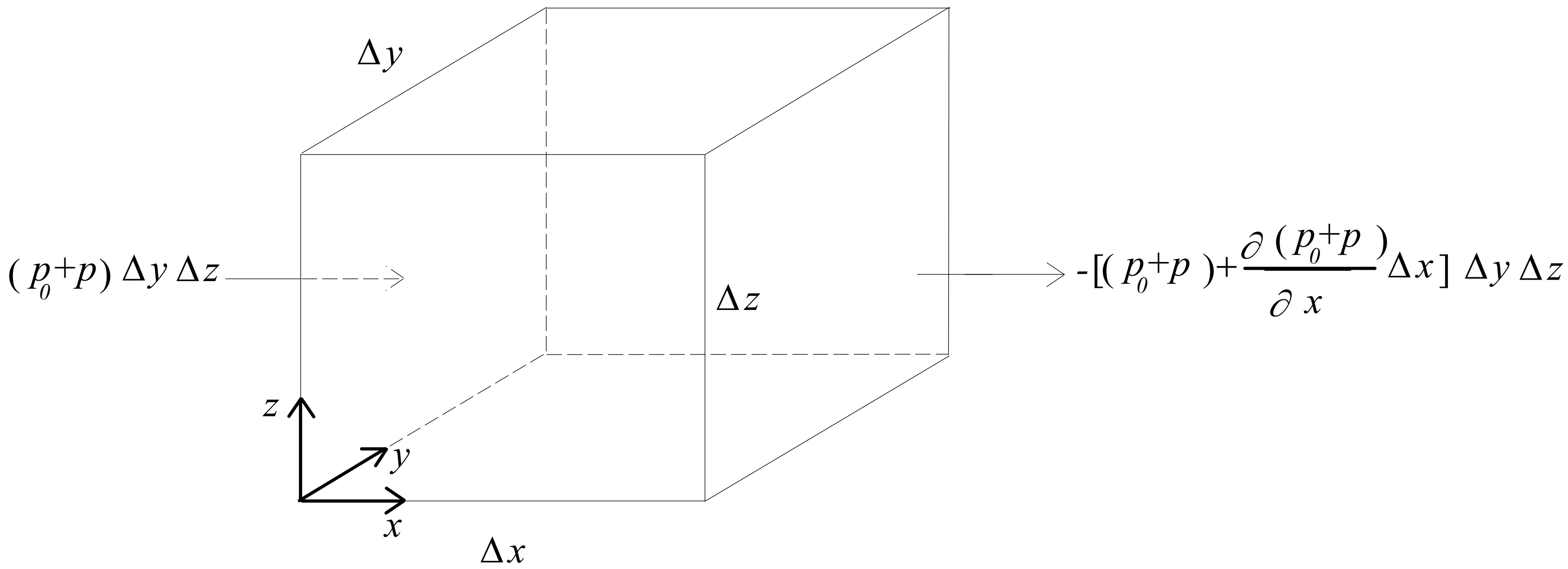
$$-\left[(\rho_t u_x \Delta y \Delta z)_x + \frac{\partial}{\partial x}(\rho_t u_x \Delta y \Delta z)_x \Delta x\right]$$

$$\frac{\partial \rho_t}{\partial t} + \frac{\partial}{\partial x}(\rho_t u_x) = 0$$

$$\rho_t(\vec{r},t) = \rho_0 + \rho(\vec{r},t)$$

$$\boxed{\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u_x}{\partial x} = 0}$$

Equation of Motion



Equation of Motion

$$F_x = \left[p_t \Delta y \Delta z - (p_t + \frac{\partial p_t}{\partial x} \Delta x) \Delta y \Delta z \right] = -\frac{\partial p_t}{\partial x} \Delta x \Delta y \Delta z$$

$$p_t(\vec{r}, t) = p_0 + p(\vec{r}, t)$$

$$F_x = -\frac{\partial p}{\partial x} \Delta x \Delta y \Delta z$$

$$a_x = \frac{\partial u_x(x,t)}{\partial t} = \frac{\partial (u_{x1}(x) u_{x2}(t))}{\partial t} = \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} \approx \frac{\partial u_x}{\partial t}$$

$$\Delta m = (\rho_\circ + \rho) \Delta V$$

$$\boxed{\rho_0 \frac{\partial u_x}{\partial t} + \frac{\partial p}{\partial x} = 0}$$

State Equation

$$(p_0 + p) = (\rho_0 + \rho)\,RT/M$$

$$\frac{(p_0 + p)}{p_0} = \left[\frac{(\rho_0 + \rho)}{\rho_0}\right]^{\gamma}$$

$$p_t = p_0 + p = p_0 + \rho \left.\frac{\partial p_t}{\partial \rho_t}\right|_{\rho_t=\rho_0} + \rho^2 \frac{1}{2} \left.\frac{\partial^2 p_t}{\partial \rho_t^2}\right|_{\rho_t=\rho_0} + \ldots$$

$$p = \rho \left.\frac{\partial p_t}{\partial \rho_t}\right|_{\rho_t=\rho_0}$$

$$\boxed{p = \beta\rho/\rho_0}$$

$$\beta = \rho_0 \left.\frac{\partial p_t}{\partial \rho_t}\right|_{\rho_t=\rho_0}$$

The 1-D Wave Equation

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u_x}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial^2 \rho}{\partial t^2} + \rho_0 \frac{\partial^2 u_x}{\partial x \partial t} = 0$$

$$\rho_0 \frac{\partial u_x}{\partial t} + \frac{\partial p}{\partial x} = 0 \quad \Rightarrow \quad \rho_0 \frac{\partial^2 u_x}{\partial x \partial t} + \frac{\partial^2 p}{\partial x^2} = 0$$

$$\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 \rho}{\partial t^2} = 0$$

$$p = \beta \rho / \rho_0 \quad \Rightarrow \quad \frac{\partial^2 p}{\partial x^2} - \frac{\rho_0}{\beta} \frac{\partial^2 p}{\partial t^2} = 0$$

$$\boxed{\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0}$$

$$c = \sqrt{\beta / \rho_0} = \sqrt{\gamma RT / M} = c_0 \sqrt{T / 273}$$

The 3-D Wave Equation

$$\boxed{\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0}$$

$$\nabla^2 p = \left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right]$$

Harmonic Solution of the Free 1-D Wave Equation

$$p(x,t) = \hat{p}_+ \cos\omega(t - x/c)$$

$$k = \omega / c$$

At a certain x_1

$$\hat{p}_+ \cos(\omega t - kx_1) = \hat{p}_+ \cos(\omega(t+T) - kx_1)$$

$$\omega = 2\pi / T$$

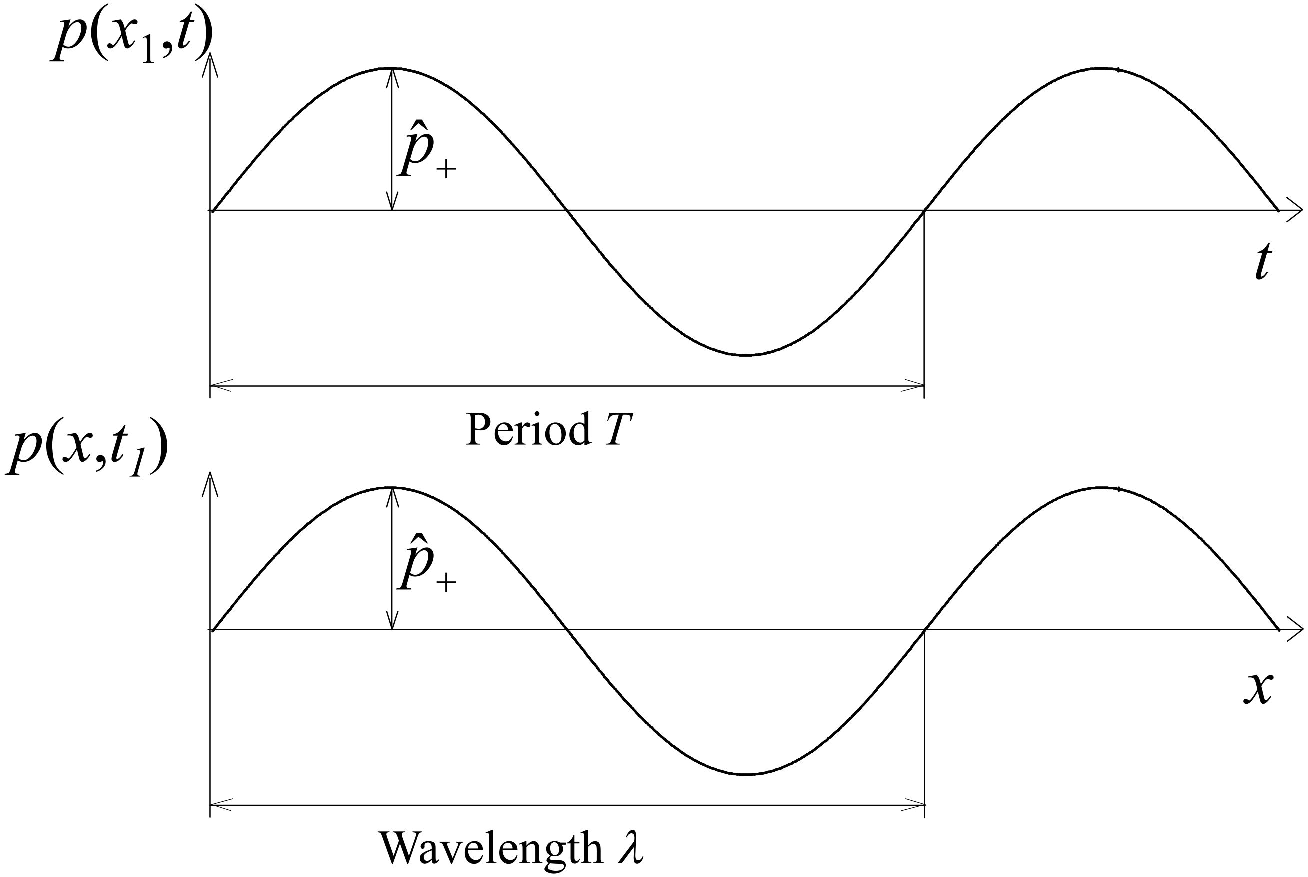
At a certain t_1

$$\hat{p}_+ \cos(\omega t_1 - kx) = \hat{p}_+ \cos\left(\omega t_1 - k\left(x + \lambda\right)\right)$$

$$k\lambda = 2\pi, k = 2\pi/\lambda$$

$$c = f\lambda$$

Harmonic Solution of the Free 1-D Wave Equation



Specific Impedance

$$Z = p/u_x$$

$$\mathbf{p}(x,t) = \hat{p}_+ e^{i(\omega t - kx)} + \hat{p}_- e^{i(\omega t + kx)}$$

$$\rho_0 \frac{\partial u_x}{\partial t} + \frac{\partial p}{\partial x} = 0 \qquad\qquad u_x = -\frac{1}{\rho_0}\int \frac{\partial p}{\partial x}\, dt$$

$$\mathbf{u}_x(x,t) = -\frac{1}{\rho_0}\left[\frac{-ik}{i\omega}\hat{p}_+ e^{i(\omega t - kx)} + \frac{ik}{i\omega}\hat{p}_- e^{i(\omega t + kx)}\right]$$

$$Z_0^+ = \rho_0 c \qquad\qquad Z_0^- = -\rho_0 c$$

Example

For air at 20° C and 1 atm $Z_0 = \rho_0 c = 415\ \mathrm{Pa{\cdot}s/m}$

Sound Intensity

$$I_x(x,t) = p(x,t)u_x(x,t)$$

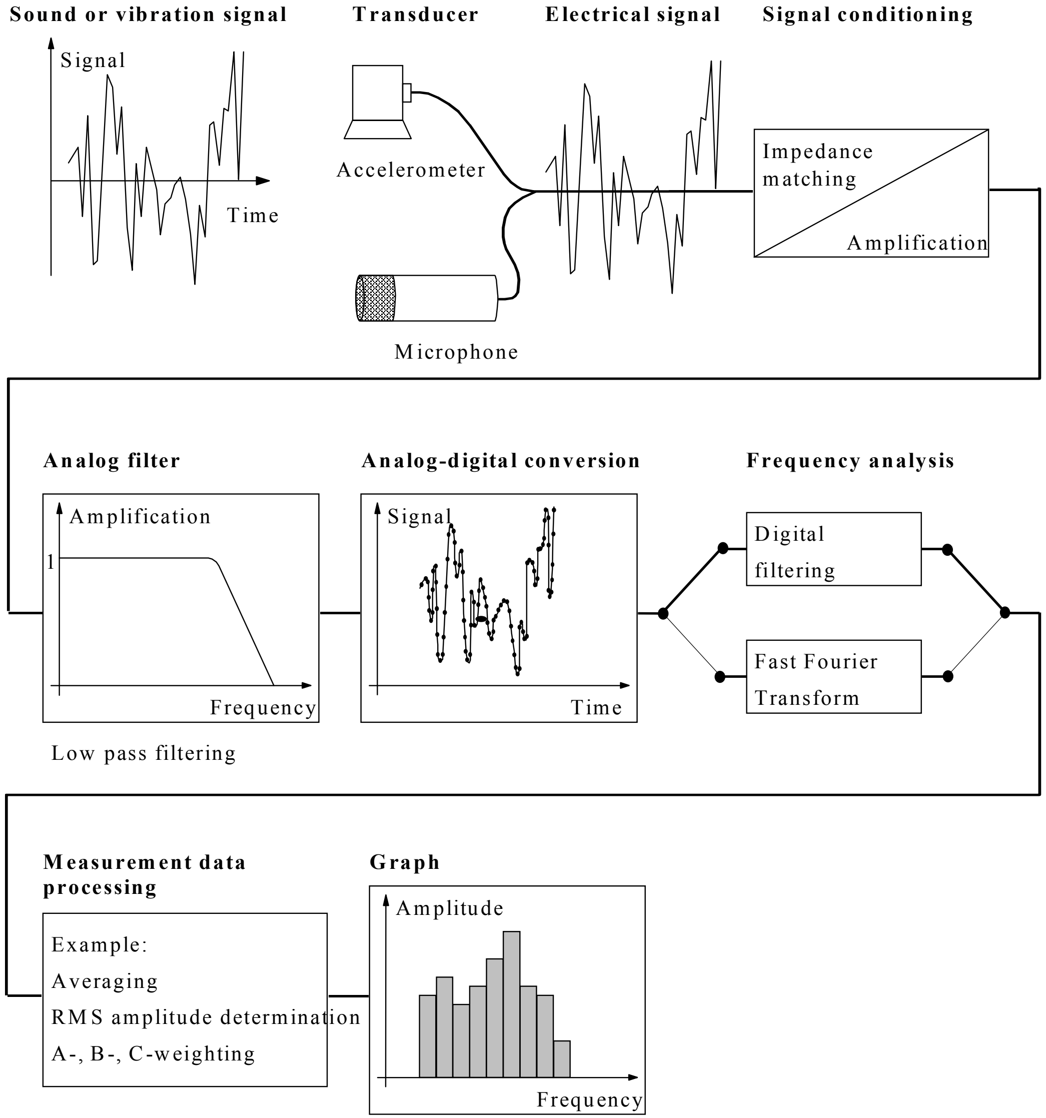
$$\bar{I}_x(x) = \frac{1}{T}\int_0^T p(x,t)u_x(x,t)dt$$

$$\bar{I}_x = (\hat{p}_+^{\;2} - \hat{p}_-^{\;2})\big/ 2\rho_0 c$$

$$\bar{I}_x = \frac{\tilde{p}_+^{\;2}}{\rho_0 c} - \frac{\tilde{p}_-^{\;2}}{\rho_0 c}$$

Sound and Vibration Measurement Techniques

Measurement Chain

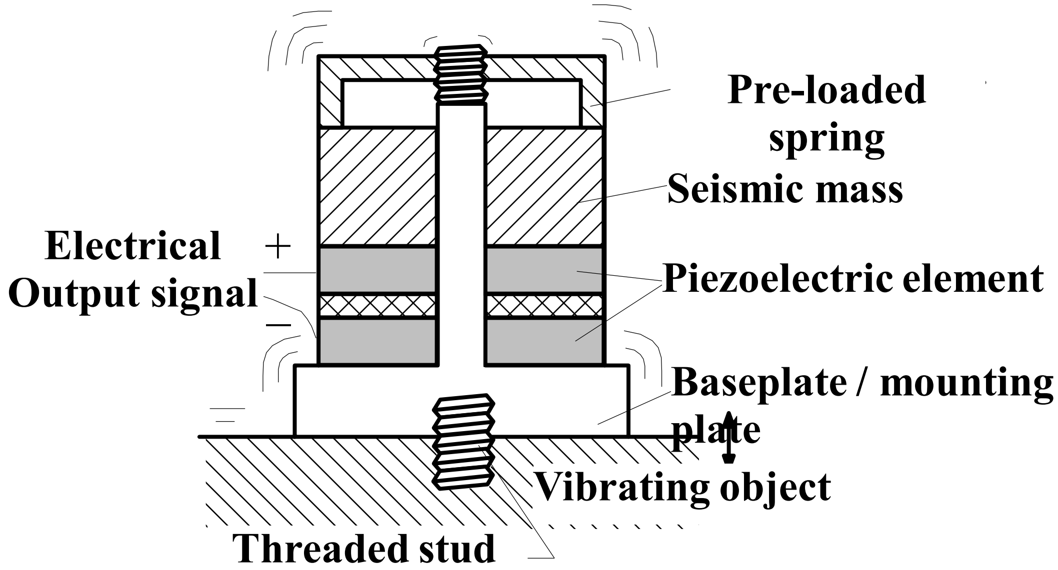


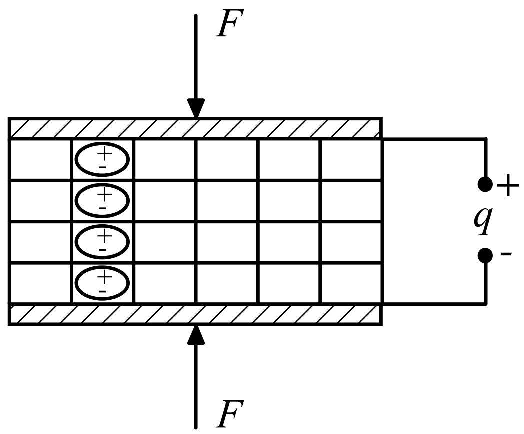
Transducers

$$U(t)[\mathrm{V}] = C\,p(t)$$

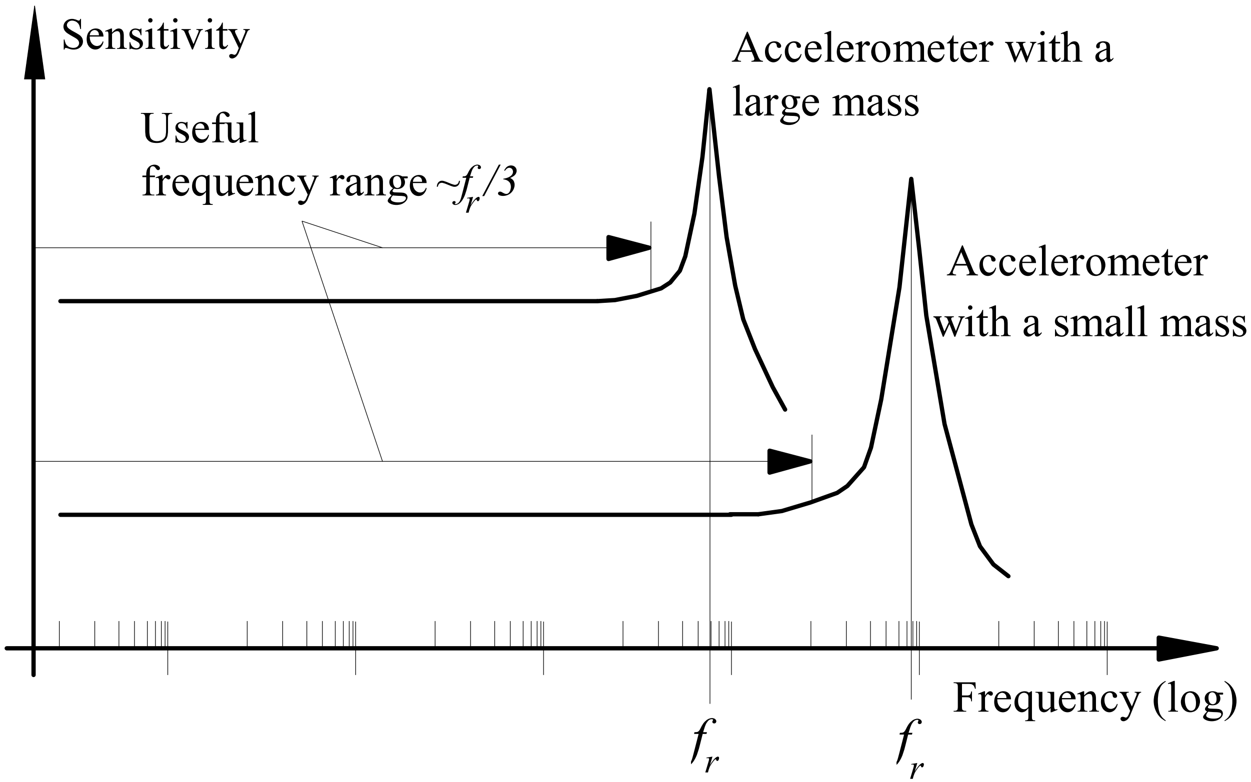
- Sensitivity
- Frequency band
- Dynamic range:

Accelerometers

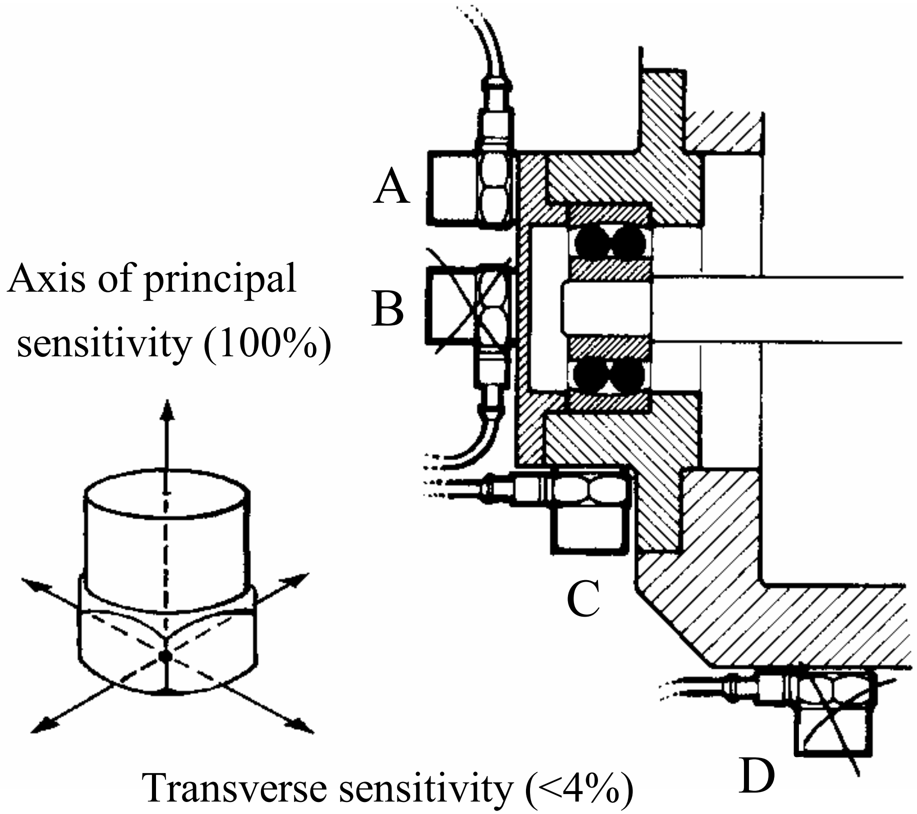




Accelerometers



Mounting of Accelerometers



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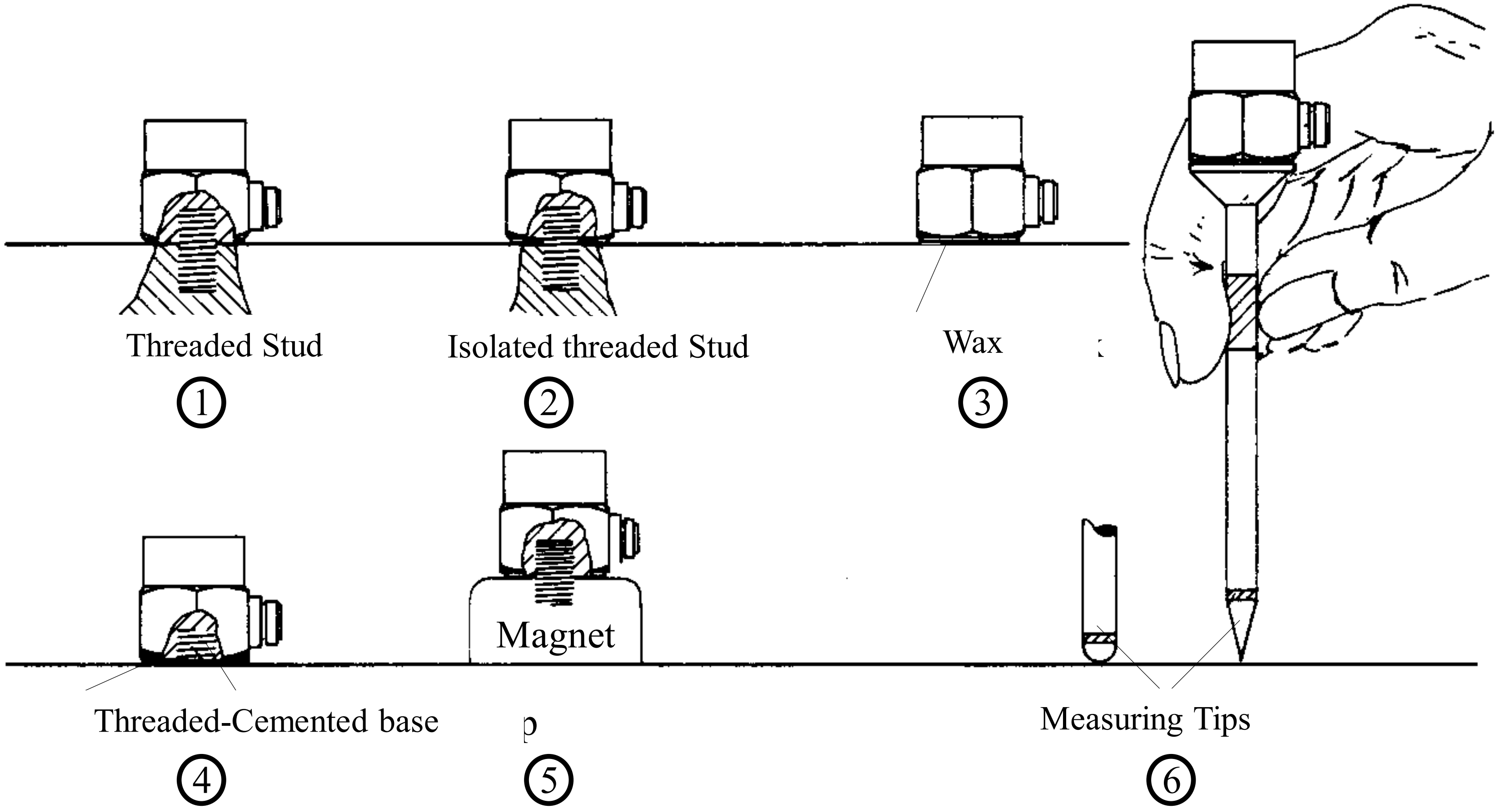
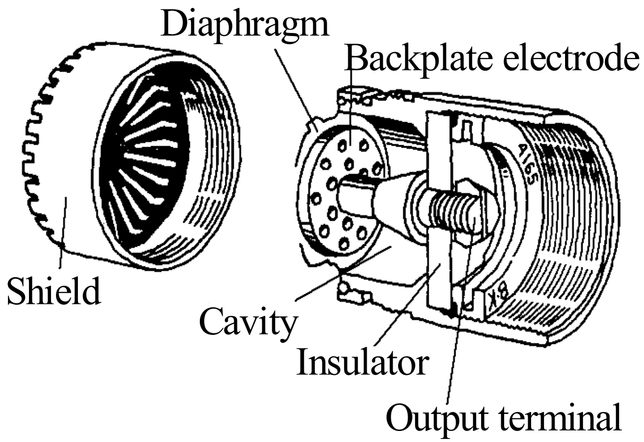
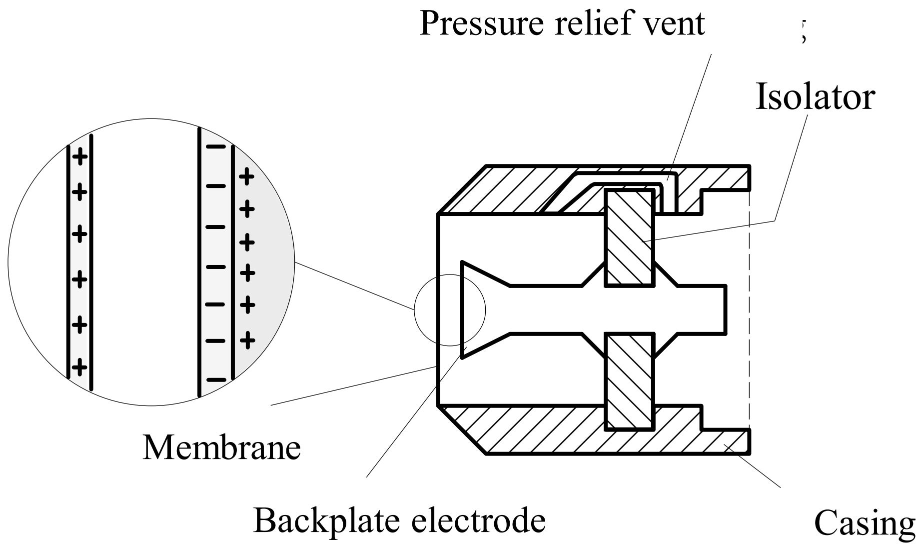


Microphones





Sound Level Meter